A cross-flow theory for the normal force on inclined bodies of revolution of large thickness ratio

By R. N. COX
Armament Research and Development Establishment, Fort Halstead, Kent

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In the Munk–Jones cross-flow theory for slender bodies of revolution (Munk 1924; Jones 1946), the cross force on an inclined body is obtained by replacing the three-dimensional flow by a non-steady two-dimensional flow, and by equating the cross-force to the rate of change of cross-flow momentum on a transverse lamina moving past the body with the free stream velocity U_0. The result obtained for the lift force L on an element of the body is, for small angles of attack α,

$$dL/dx = \tfrac{1}{2}\rho U_0^2 (dA/dx)\, 2\alpha, \tag{1}$$

where A is the cross-sectional area of the body, and, by integration

$$L = \tfrac{1}{2}\rho U_0^2 A_B\, 2\alpha, \tag{1a}$$

where A_B is the base area of the body.

Although the fact is not generally recognized, this result is in poor agreement with exact calculations and with experimental data for finite bodies of revolution, and in order to develop a cross-flow theory for bodies of large thickness ratio, it is necessary to examine the assumptions of the simple theory.

The main assumption is that the flow in the cross-plane is approximately two-dimensional. Thus, in the three-dimensional incompressible potential equation (using wind axes)

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \tag{2}$$

the first term must be small compared with the other two. By writing $x = lX$, $y = sY$, $z = sZ$, where s is a typical width and l a typical length of the body, and $\phi(x,y,z) = U_0\, l\Phi(X, Y, Z)$, we may re-write (2) as

$$\frac{s^2}{l^2}\Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} = 0. \tag{3}$$

It is evident that the condition that the first term is small is that $s^2/l^2 \ll 1$. For compressible flows satisfying the Prandtl–Glauert differential equation for the perturbation potential:

$$\beta^2\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{4}$$

where $\beta = (1-M^2)^{1/2}$, the compressible flow is related to the incompressible flow past a body of revolution elongated in the x-direction by a factor β, and the condition for two-dimensional flow in the cross-plane becomes, for the compressible flow, $\beta^2 s^2/l^2 \ll 1$. For moderate values of β, this condition will hold for surprisingly large values of the body thickness ratio. (For a 20° cone, for instance, $s^2/l^2 \doteqdot 0{\cdot}1$.)

In order to calculate the cross-flow, it is necessary to solve the two-dimensional equation

$$\phi_{yy} + \phi_{zz} = 0 \tag{5}$$

with ϕ_n specified on the boundary. In the Munk–Jones approximation, this boundary condition becomes (using cylindrical coordinates (x, r, ω) with the body shape defined by $R(x)$)

$$\phi_r(x, R, \omega) = U_0 \cos\alpha \frac{d}{dx} R(x) + U_0 \sin\alpha \cos\omega, \tag{6}$$

or, by superposition for the cross-flow, writing $\phi_r = \phi_{1r} + \phi_{2r}$,

$$\phi_{2r}(x, R, \omega) = U_0 \alpha \cos\omega \tag{7}$$

for small angles of attack.

This approximate tangency condition amounts to an assumption that the longitudinal component of the velocity on the body, which is the velocity with which the transverse lamina moves along the body, is approximately equal to the main stream velocity U_0. This is manifestly untrue for thick bodies of revolution. For a 20° cone with a free stream Mach number of 1·3, for instance, the surface velocity is $0{\cdot}8 U_0$, and the longitudinal component is $0{\cdot}75 U_0$. This suggests that an improvement to the cross-flow theory may be made by employing a more exact tangency condition in which the main stream velocity is replaced by the longitudinal component of the surface velocity $U_s(x)$. The lamina then moves along the body with a velocity $U_s \cos\theta$, where θ is the inclination of the surface to the body axis.

Thus (7) becomes

$$\phi_{2r}(x, R, \omega) = U_s \cos\omega\, \alpha. \tag{8}$$

The lift force on an element of the body is then

$$dL/dx = \tfrac{1}{2}\rho U_s^2 \cos^2\theta (dA/dx)\, 2\alpha. \tag{9}$$

In order to illustrate the use of this result, a comparison is made in figure 1 with the exact calculation by Kopal (1947) of the supersonic flow past yawed cones of semi-angle θ_0. Since, for such flows, the surface velocity U_s is constant, equation (9) may be integrated to yield

$$L = \tfrac{1}{2}\rho U_0^2 \left(\frac{U_s \cos\theta_0}{U_0}\right)^2 A_B\, 2\alpha. \tag{10}$$

The lift coefficient slope is then

$$[C_{L_\alpha}]_{\alpha=0} = \frac{L}{\tfrac{1}{2}\rho U_0^2 A_B \alpha} = 2\left(\frac{U_s \cos\theta_0}{U_0}\right)^2,$$

where U_s is obtained from Kopal's calculations. To convert the quantities tabulated by Kopal to the notation used here, the following relationships were employed:

$$U_s = \overline{U}_s\{v^2 + 2a_1^2/(\gamma - 1)\}^{1/2}, \quad \text{and} \quad C_{L_\alpha} = C_{N_\alpha} - C_D,$$

where $$C_{N_\alpha} = 8K_N/\pi, \qquad C_D = 8K_D/\pi,$$

The results given in figure 1 for cones of semi-angle 10°, 20° and 30° show that the present method gives good agreement with the exact results for cones up to 30° semi-angle, and correctly predicts the variation of C_{L_α} with Mach number for quite large Mach numbers. This implies that the cross-flow close to the body remains approximately two-dimensional even for cases in which the parameter $\beta^2 s^2/l^2$ becomes large. The Munk–Jones value $C_{L_\alpha} = 2(\text{radian})^{-1}$ is also shown on the figure.

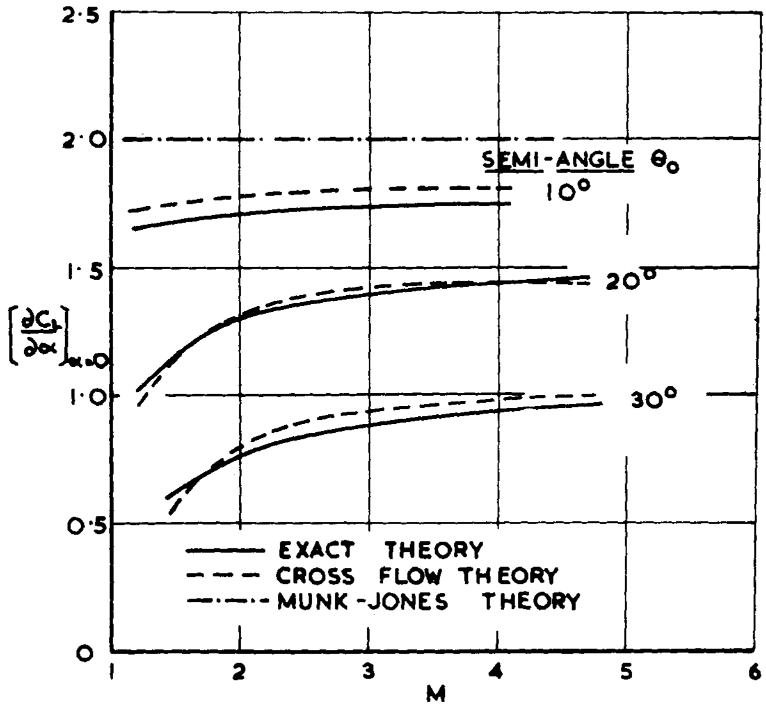

Figure 1. Variation of lift force slope with Mach number for supersonic flow past inclined cones.

The method has also been employed by the author (Cox & Maccoll 1956) for calculating the cross-force on yawed cones in fully developed cavitating flow.

References

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